# Extended-body effects in general relativity
## *What is possible?*


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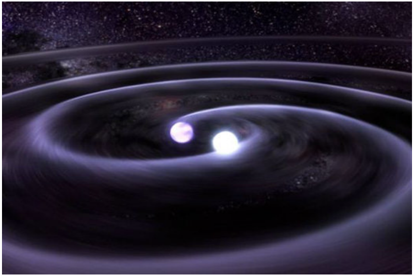

How do things fall?

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1. **Newtonian gravity**: There's a gravitational potential $\phi$ and each object accelerates via $\ddot{\boldsymbol{z}} = -\boldsymbol{\nabla}\phi$.

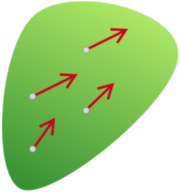
2. **General relativity**: There's a metric $g_{ab}$ which determines $\nabla_a[g]$. Each object moves on a geodesic via $\dot{z}^b \nabla_b[g] \dot{z}^a = 0$.

**Equivalence principle**: Free-fall is universal

# Is that *really* how things fall?

In Newtonian gravity,

$$\ddot{\boldsymbol{z}} = -\int (\rho/m)\boldsymbol{\nabla}\phi dV.$$



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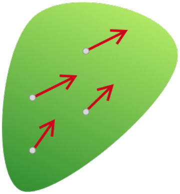

When is this *exactly* equal to $-\boldsymbol{\nabla}\phi$?

...at least when bodies are spherical.

# Corrections to spherical-body motion

$$\ddot{z}_i = -\nabla_i \phi + (?)$$

Two types of corrections:

1. Self-interaction requires that $\phi$ be replaced by $\phi_{\mathrm{ext}}$.
2. Extended-body effects bring in aspects of the internal structure: Free-fall is no longer universal.

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$$\ddot{z}_i = \underbrace{-\nabla_i \phi_{\rm ext}}_{\text{monopole}} - \underbrace{\tfrac{1}{2}(\overbrace{Q^{jk}}^{\text{(quadrupole moment)}}/m)\nabla_i \nabla_j \nabla_k \phi_{\rm ext}}_{\text{quadrupole}} + \ldots$$

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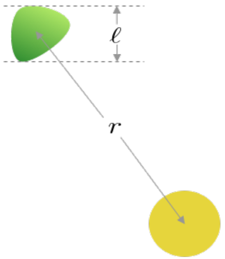
$$\ddot{z}_i = -\nabla_i \phi_{\text{ext}} - \underbrace{\tfrac{1}{2}(Q^{jk}/m)\nabla_i\nabla_j\nabla_k\phi_{\text{ext}}}_{\mathscr{O}[(\ell/r)^2 \times \text{(monopole)}]} + \ldots$$




At leading nontrivial order, effects of internal structure are determined by the **quadrupole moment** $Q^{ij} \sim m\ell^2$.

# Rocket-free maneuvering?

That which is not forbidden is allowed. . .

. . . and maneuvers are forbidden mainly by symmetries.

# Examples of Newtonian maneuvering [AIH & Gaffney (2020)]

1. **Homogeneous gravitational field**: 6 symmetries fix 6 force and torque components $\Rightarrow$ no maneuvering possible

2. **Spherically-symmetric field**: 3 symmetries $\Rightarrow$ 3 possible force and torque components ($\boldsymbol{L} + \boldsymbol{S} = \mathrm{const}$).

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You need *gradients* to "grab onto..."

# Motion in GR?

Extended-body effects derived in full by Dixon (1974), at least without self-interaction. Self-interaction incorporated later AIH (2012, 2015).

$$\dot{p}_a = -\frac{1}{2} R_{abcd} \dot{z}^b S^{cd} + \text{(int struct)},$$
$$\dot{S}_{ab} = 2 p_{[a} \dot{z}_{b]} + \text{(int struct)}.$$

There can also be a nontrivial momentum-velocity relation:

$$p^a = m \dot{z}^a + \text{(hidden momentum)}$$

# Universality of free-fall in GR

There are spin effects, but these aren't controllable.

First non-universal contributions to motion are from the quadrupole moment. This has 10 components in vacuum spacetimes.

# Irrelevant quadrupole components

Not all 10 quadrupole components can affect motion [AIH, 2020]

1. Type D spacetimes:
   - At least 4 components are irrelevant.
   - In *Schwarzschild*, 5 components are irrelevant.
2. Type N:
   - At least 6 components are irrelevant.
   - For *linearly-polarized plane waves*, 9 are irrelevant.
3. ...

# Rocket-free maneuvers in Schwarzschild [AIH, 2020]

Suppose that:

1. Quadrupole is controlled so torque vanishes.
2. Remainder of quadrupole is used to control motion

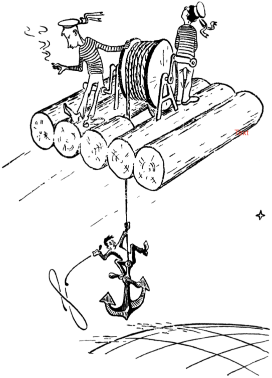
5 quadrupole components are irrelevant. 4 must be controlled to control the torque. There is 1 left over. . . What can this do?

# Everything is in the mass

Extended-body effects enter *only* via changes in mass:

$$dm = -\mathscr{J}\, d(M/r^3).$$

$E$ and $\vec{J}$ are conserved. $E/m$ and $\vec{J}/m$ are not.



[Beletsky, 2001]

# Net mass changes

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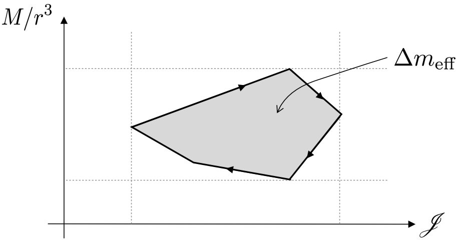
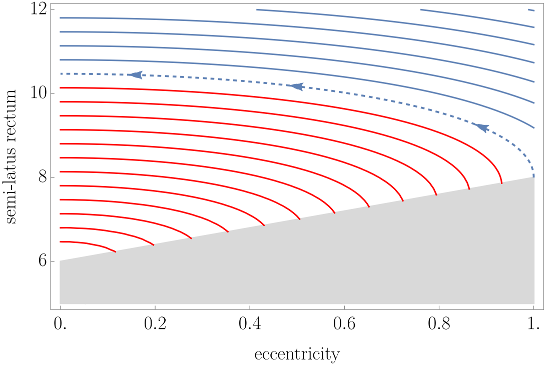
$$dm = -\mathscr{J}\, d(M/r^3), \qquad d(\overbrace{m + \underbrace{\mathscr{J} M/r^3}_{\text{“}-U_{\text{quad}}\text{”}}}^{m_{\text{eff}}}) = (M/r^3) d\mathscr{J}$$



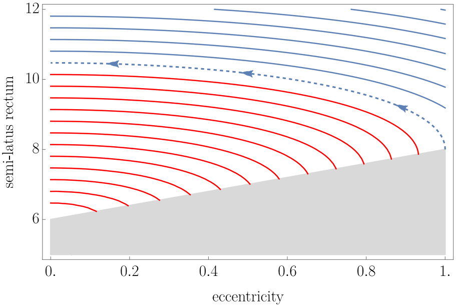

Repeating cycles with **nonzero area** raises or lowers the mass.

# What do mass changes accomplish?




- Increasing mass $\Rightarrow$ circularization
- Decreasing mass is more complicated...

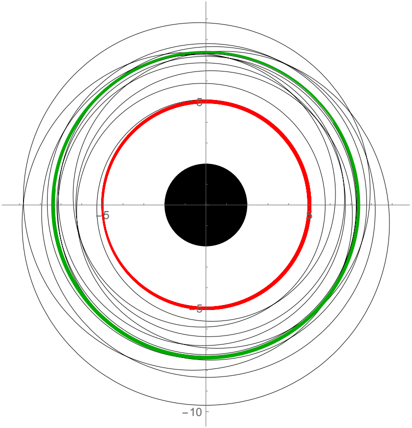
# Effect of decreasing mass

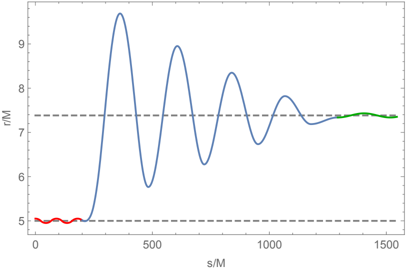



1. If $r_i > 2(3 + \sqrt{5})M \approx 10.5M$, you can escape!
2. Otherwise, decreasing mass results in an approach to an unstable circular orbit, or a plunge.

# Circular transfers

Move between pairs of circular geodesics with $4 < r/M < 10.5$.



# Relativistic vs. Newtonian effects

## [AIH (2020) vs. AIH & Gaffney (2020)]

In the spin-free, torque-free context, they're almost identical:

| Concept | Relativistic | Newtonian |
|---|---|---|
| Torque-free force | $\nabla(\mathscr{I} M/r^3)$ | $\nabla(\mathscr{I} M/r^3)$ |
| Always constant | $E$, $\vec{L}$ | $m_N$, $\vec{L}$ |
| Const when $\mathscr{I}$ jumps | $m$ | $E_{\mathrm{pt}}$ |
| Const when $\mathscr{I} = \mathrm{const}$ | $m_{\mathrm{eff}}$ | $E_{\mathrm{pt}} + U_{\mathrm{ext}}$ |

$$E - m \sim E_{\mathrm{pt}}, \qquad E - m_{\mathrm{eff}} \sim E_{\mathrm{pt}} + U_{\mathrm{ext}}$$

# Conclusions

1. Extended-body effects allow objects to control their motion.
2. They can change mass, allowing for bound orbits to be unbound, for unstable orbits to be stabilized, and more.

Even more is possible with spin and torque... There are many options for those who wait!