Metric-independence of electromagnetic fields

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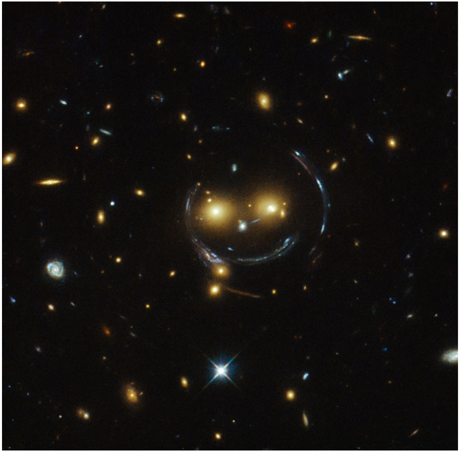
Understanding general relativity

1. How does geometry affect matter?
2. How does matter affect geometry?

One of the simplest types of “matter” is light. . .

How is electromagnetism affected by gravity?

Gravity does appear to affect light. . .



Lots of known effects

1. Gravitational lensing: light deflection, intensity modulation, time delays, redshifts, etc.

2. Gravitational waves can be detected using interferometers which employ circulating laser light, or (in principle) by timing radio signals from pulsars.

It looks like much can be learned about spacetime from the properties of electromagnetic fields. . .

All of these effects technically involve not only interactions between gravity and light, but also between light and "normal" matter.

How much is *intrinsically* electromagnetic?

A precise question

Given an electromagnetic field F_{ab} known to be compatible with a metric g_{ab}, are there other metrics g'_{ab} with which it is also compatible?

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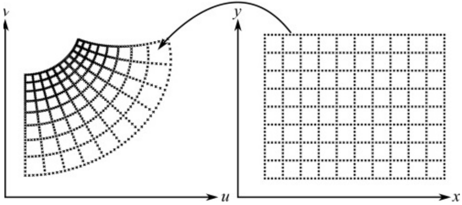
Yes, conformal transformations preserve vacuum electromagnetic fields: $g_{ab} \mapsto g'_{ab} = \Omega^2 g_{ab}$ for any $\Omega^2 > 0$ [Bateman & Cunningham, 1910].

Geometrically, conformal transformations preserve angles and local causality while rescaling lengths.



EM fields are locally scale-invariant. . .

A simple application

Many results from flat-spacetime electromagnetism—which we understand very well—hold *without change* in certain curved spacetimes:

1. (Anti-) de Sitter
2. Friedman-Robertson-Walker cosmologies
3. Special gravitoelectromagnetic plane waves

One need only identify a flat metric η_{ab} for which $g_{ab} = \Omega^2 \eta_{ab}$...

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- Given one (perhaps flat) metric, not many physically interesting metrics are conformally related.
- "Most" of the physical content of a metric is not conformal.

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Is the conformal ambiguity all there is?

Maxwell's equations

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$$g^{ab}\nabla_a F_{bc} = 0, \qquad \nabla_{[a}F_{bc]} = 0,$$

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But these can be recast as

$$\mathrm{d}{*}\boldsymbol{F} = 0, \qquad \mathrm{d}\boldsymbol{F} = 0,$$

using the language of differential forms

Maxwell's equations and Hodge duals

The metric enters only via the Hodge dual

$$F_{ab} \mapsto (*F)_{ab} = \frac{1}{2}\epsilon_{abcd} g^{ce} g^{df} F_{ef} ,$$

which is purely algebraic.

If two metrics g_{ab} and g'_{ab} give the same $*$ operator for all 2-forms in 4 dimensions, Maxwell solutions locally remain Maxwell solutions.

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But this happens only for conformally-related metrics [Dray, 1989]...

Change the demand a bit:

If two metrics g_{ab} and g'_{ab} give the same $*$ operator for ~~*all*~~ a particular 2-form in 4 dimensions, that Maxwell solution locally remains a Maxwell solution.

Now there are many more possibilities...

Adding matter

Preserving $*\boldsymbol{F}$ preserves solutions to Maxwell's equations not only in vacuum, but also in force-free plasmas: black hole accretion disks, pulsar magnetosphere, solar corona, fusion reactors, ...

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But this simplifies to

$$\mathrm{d}\alpha \wedge \mathrm{d}{*\boldsymbol{F}} = 0, \qquad \mathrm{d}\beta \wedge \mathrm{d}{*\boldsymbol{F}} = 0$$

with $\boldsymbol{F} = \mathrm{d}\alpha \wedge \mathrm{d}\beta$ [Gralla & Jacobson, 2014].

Given some (g_{ab}, F_{ab}), find all metrics g'_{ab} such that $*_g \boldsymbol{F} = *_{g'} \boldsymbol{F}$.

Just an exercise in linear algebra. . .

Solution depends on eigenvectors and eigenvalues of F_{ab}. . .

Two types of solutions, depending on

$$|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2 = \frac{1}{2} F^{ab} F_{ab}, \qquad \boldsymbol{E} \cdot \boldsymbol{B} = \frac{1}{2} F^{ab} * F_{ab}.$$

1. $|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2 = \boldsymbol{E} \cdot \boldsymbol{B} = 0$ implies that F_{ab} is null.
    - Plane waves
    - Geometric optics
    - *Not* generic
2. Otherwise F_{ab} is non-null.
    - Generic case

Principal null directions

It is useful to characterize F_{ab} by its *null* eigenvectors (with respect to g_{ab}). These satisfy

$$g^{ab}\ell_a F_{b[c}\ell_{d]} = 0, \qquad g^{ab}\ell_a \ell_b = 0,$$

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1. Real null $F_{ab} \Rightarrow$ 1 real PND,
2. Real non-null $F_{ab} \Rightarrow$ 2 real and 2 complex PNDs.

Null case

If F_{ab} is a null solution to the vacuum or force-free Maxwell equations with metric g_{ab} and associated null eigenvector ℓ_a, it is also a solution with metrics

$$g'_{ab} = \Omega^2(g_{ab} + \ell_{(a}\xi_{b)}),$$

for arbitrary Ω and ξ_a.

Five free functions!

This also works for trajectories in geometric optics: Null geodesic congruences with tangent ℓ^a remain null geodesic congruences.

A special case

If $\xi_a \propto \ell_a$, the general transformation reduces to

$$g' = [(\text{conformal xform}) \circ (\text{Kerr-Schild xform})](g).$$

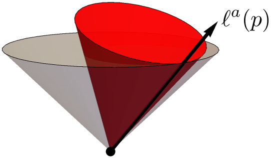
1. Physically important.
2. Also works for non-null fields!

Kerr-Schild transformations

Deform $g_{ab} \mapsto g'_{ab}$ using a *null* 1-form ℓ_a:
[Trautman, 1962; Kerr & Schild, 1965]

$$g'_{ab} = g_{ab} + V\ell_a\ell_b.$$

1. Geometrically, this deforms light cones: Rays tangent to ℓ^a are preserved while others change.



2. Algebraically, $h_{ab} = V\ell_a\ell_b$ is a square root of zero:

$$h_{ab}h^b{}_c = 0.$$

Physical significance of Kerr-Schild

1. Some of the most important metrics are KS deformations of flat spacetime: Schwarzschild, Kerr, plane waves, ...
2. Stationary, spherically-symmetric metrics are all conformal and KS xforms applied to flat spacetime [Mitskievich & Horský, 1996]
3. Effectively linearizes Einstein's equation [Gürses & Gürsey, 1975; Xanthopoulos, 1978]
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KS also interacts nicely with Maxwell's equations!

General metric transformation law for a *non-null* F_{ab} with null eigenvectors ℓ_a, k_a, m_a and $\bar{m}_a$ has the form

$$g' = (\mathrm{Conf} \circ \mathrm{KS}_\ell \circ \mathrm{KS}_k \circ \mathrm{KS}_m \circ \mathrm{KS}_{\bar{m}})(g).$$

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Again, **five free functions**!

For arbitrary real Ω, V, W and complex Y,

$$g'_{ab} = \Omega^2 \left[g_{ab} + \frac{V\ell_a\ell_b + Wk_ak_b + (k\cdot\ell)VW\ell_{(a}k_{b)}}{1 - \frac{1}{4}(k\cdot\ell)^2 VW} + \frac{Ym_am_b + \bar{Y}\bar{m}_a\bar{m}_b + (m\cdot\bar{m})|Y|^2 m_{(a}\bar{m}_{b)}}{1 - \frac{1}{4}(m\cdot\bar{m})^2|Y|^2} \right].$$

Summary

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What does this imply?

Physical interpretation

Some physical properties of EM fields are changed only by rescalings, or not at all:

$$|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2 \mapsto \Omega^{-4}(|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2)$$

$$\boldsymbol{E} \cdot \boldsymbol{B} \mapsto \Omega^{-4}(\boldsymbol{E} \cdot \boldsymbol{B})$$

$$T^a{}_b \mapsto \Omega^{-4}(\ldots) T^a{}_b$$

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But some things change completely...

An example: Gravitational waves

Plane-fronted gravitational waves with parallel rays (*pp*-waves):

$$g'_{ab} = \eta_{ab} + H\ell_a \ell_b, \qquad \mathcal{L}_\ell H = 0.$$

Flat-spacetime solutions with PND ℓ_a, such as

$$\boldsymbol{F} = \mathrm{d}\phi \wedge \boldsymbol{\ell}, \qquad \mathcal{L}_\ell \phi = 0,$$

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EM waves propagate identically, *regardless of the gravitational waveform*, determined by $H(x)$, *or the electromagnetic waveform*, determined by $\phi(x)$.

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In TT-gauge where

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + h_{ij} dx^i dx^j + O(h^2),$$

coordinate components $F_{\mu\nu}$ depend on h_{ij}!

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- Physically, the coordinates are constructed using timelike geodesics, and these are not preserved by the metric xform.
- Field components seen by freely-falling observers *do change*...

Lesson #1

Physical interpretations can be subtle.

Lesson #2

Interesting metrics can be generated by the allowed transformations, but often in non-obvious ways.

Example 2: Plane waves to spherical waves

Cartesian $\to$ spherical xform in flat spacetime:

$$\boldsymbol{F} = \mathrm{d}\phi(t - z, x, y) \wedge \mathrm{d}(t - z) \mapsto \mathrm{d}\Phi(t - r, \theta, \varphi) \wedge \mathrm{d}(t - r)$$

Transforming again makes these valid in **all** spherically-symmetric, stationary (and many non-stationary) spacetimes. . .

Mapping back to flat spacetime explains lack of backscattering seen in force-free solutions around black holes [Brennan, Gralla, & Jacobson (2013)].

Given metric transformation rules guarantee $*F \mapsto *F$.

But all that matters is that $\mathrm{d} * F$ is preserved, up to scaling. Suppose

$$*\boldsymbol{F} \mapsto \gamma(*\boldsymbol{F}) + d\lambda, \qquad \mathrm{d}\gamma \wedge *F = 0.$$

Previously assumed $\mathrm{d}\lambda = 0$ and $\gamma = 1$; more is possible!

And even more if $(\boldsymbol{F}, g_{ab}) \mapsto (\boldsymbol{F}', g'_{ab})\ldots$

A gravitational analog?

Given a 1st-order metric perturbation h_{ab} compatible for a given background g_{ab}, is it compatible with another background g'_{ab}?

True at least for Kerr-Schild perturbations: If

$$h_{ab} = V\ell_a\ell_b, \qquad g^{ab}\ell_a\ell_b = 0,$$

one can show that $g'_{ab} = g_{ab} + W\ell_a\ell_b$.

This probably generalizes...

Conclusions

1. It can be interesting to find transformations which preserve *specific* solutions, rather than *all* solutions.
2. For Maxwell's equations, each solution is compatible with at least five free functions worth of metrics: Not much matters!
3. $\Rightarrow$ New method to generate and interpret EM fields.

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1. General interpretation of metric xforms?
2. Can electrodynamics be understood "modulo" all of this?
3. Other types of fields?